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Dielectric Tensor for a Quantum Plasma*

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The dispersive properties of a plasma in a uniform magnetic field are investigated using a quantum "distribution function" (Wigner function). The Wigner function is shown to satisfy the Boltzmann-Vlasov equation in the long-wavelength approximation. The dielectric tensor is evaluated at $T=0°K$ for waves propagating along and across the magnetic field. It is conjectured that a simple "quantized kinetic theory" is applicable in areas previously explored only by more sophisticated means.

I. INTRODUCTION

THE dispersive character of a quantum plasma has been studied by numerous workers.[1–13] Lindhard[4] investigated the response of a quantum plasma to transverse and longitudinal waves using a self-consistent field procedure. Cohen and Ehrenreich[5] illuminated the connection between the self-consistent field approach and the second quantization formalism of Sawada[9] and Brout.[10] The effects of spin and exchange have been considered by Burt and Wahlquist.[11] Calculations for a quantum plasma in a uniform magnetic field have been performed by Quinn and Rodriguez,[12] using a density matrix formulation, and by Stephen,[13] using a field-theoretic technique. It is to be expected that our results will be identical with those of Refs. 12 and 13.

The use of the self-consistent field in plasma physics may be traced back at least as far as the work of Langmuir and Tonks.[14] The merger of kinetic theory and the self-consistent field brought about by Vlasov[15] has been extended by others, notably Gross[16] and Bernstein,[17] to include the effects of an external magnetic field. Accounts of the classical theory may be found in the reviews of Oster[18] and Bernstern and Trehan,[19] and in the recent monograph by Stix.[20]

The classical analysis is based on solutions of the Boltzmann-Vlasov equation

$$\frac{\partial f}{\partial t}+\frac{\mathbf{p}}{M}\cdot\nabla f+\mathbf{F}\cdot\nabla_p f=0. \qquad (1)$$

$f(\mathbf{r},\mathbf{p},t)$ is the distribution function and $\mathbf{F}$ denotes the total force exerted by the external and self-consistent fields. The solution of this equation for $f(\mathbf{r},\mathbf{p},t)$ allows one to calculate the relevant kinetic averages, viz., the current density

$$\mathbf{J}(\mathbf{r},t)=\frac{e}{M}\int \mathbf{p}f(\mathbf{r},\mathbf{p},t)d^3p. \qquad (2)$$

This in turn leads to the electrical conductivity tensor or, equivalently, to the dielectric tensor, and the dispersion relation.

Because of the physical transparency of the distribution function method it is desirable to develop the quantum theory in a parallel fashion. There is no unique way to proceed, for no precise quantum counterpart of the distribution function can exist. The most frequently used quantum distribution function is the so-called Wigner function.[21] Let $\rho(\mathbf{r}',\mathbf{r}'',t)$ denote the coordinate representation of the single-particle density matrix. The single-particle Wigner function, as customarily defined, is

$$f(\mathbf{r},\mathbf{p},t)=(\pi\hbar)^{-3}\int \exp(2i\mathbf{p}\cdot\boldsymbol{\lambda}/\hbar)\rho(\mathbf{r}-\boldsymbol{\lambda},\,\mathbf{r}+\boldsymbol{\lambda},\,t)d^3\lambda. \qquad (3)$$

The Wigner function allows one to calculate kinetic averages via the classical recipe. For example, the current density is given by (2). Unlike the classical distribution function, however, the Wigner function can not be interpreted as a probability density in phase space. The correlation between position and momentum engendered by the uncertainty principle permit the Wigner function to assume *negative* values, ruling out any interpretation as a probability density.[22]

After languishing in unexploited obscurity for twenty years the Wigner function is finding increased favor among theoreticians.[23–26] Von Roos[6,27] and his co-

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workers[11,28] have used a slightly different quantum distribution function which has essentially the same properties as the Wigner function in so far as the calculation of kinetic averages is concerned. We choose to work with the Wigner function because it most closely corresponds to the classical distribution function.

In Sec. III we discuss the plasma model and self-consistent field. A modified Wigner function, suitable for describing a plasma in an electromagnetic field, is introduced. It is shown that this Wigner function satisfies the Boltzmann-Vlasov equation in the long-wavelength approximation. The terms neglected by such an approximation are of the same order as those ignored by omitting the interaction of the electron magnetic moment and the *self-consistent* magnetic field.

In Sec. III the Fourier transform of the linearized equation is solved and a formal expression is obtained for the dielectric tensor. The equilibrium Wigner function is introduced in Sec. IV for Boltzmann and Fermi-Dirac statistics. The dielectric tensor is evaluated in Sec. V for waves propagating along and across the magnetic field.

II. WIGNER FUNCTION; KINETIC EQUATION

We study the customary plasma model—an electron gas of infinite extent neutralized by a uniform background of positive charge. The plasma is permeated by a uniform magnetic field ($\mathbf{B}$) whose direction defines the z axis of a Cartesian coordinate system. The external field is conveniently represented by a vector potential

$$\mathbf{A}_0(\mathbf{r})=\tfrac{1}{2}\mathbf{B}\times\mathbf{r};\quad \mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}_0. \tag{4}$$

In addition to the external field, each electron is assumed to be acted upon by a self-consistent electromagnetic field specified by the scalar and vector potentials $\varphi(\mathbf{r},t)$ and $\mathbf{A}_1(\mathbf{r},t)$. The self-consistent electric and magnetic fields are given by

$$\mathbf{E}_1=-\frac{1}{c}\frac{\partial\mathbf{A}_1}{\partial t}-\boldsymbol{\nabla}\varphi. \tag{5}$$

$$\mathbf{B}_1=\boldsymbol{\nabla}\times\mathbf{A}_1, \tag{6}$$

and φ and $\mathbf{A}_1$ are related by the Lorentz gauge condition

$$\boldsymbol{\nabla}\cdot\mathbf{A}_1+\frac{1}{c}\frac{\partial\varphi}{\partial t}=0. \tag{7}$$

In situations where it is not necessary to distinguish between $\mathbf{A}_0$ and $\mathbf{A}_1$ we write the total vector potential as $\mathbf{A}$,

$$\mathbf{A}=\mathbf{A}_0+\mathbf{A}_1. \tag{8}$$

The Hamiltonian for an electron (charge e, mass M) may be written

$$\hat{H}=\frac{1}{2M}\left(-i\hbar\boldsymbol{\nabla}-\frac{e}{c}\mathbf{A}\right)^2+e\varphi-\hat{\mu}\cdot(\mathbf{B}+\mathbf{B}_1), \tag{9}$$

where $\hat{\mu}$ is the magnetic moment operator for the electron. The single-particle density matrix $\rho(\mathbf{r}',\mathbf{r}'',t)$ satisfies the equation

$$i\hbar\frac{\partial\rho}{\partial t}=(\hat{H}(\mathbf{r}')-\hat{H}(\mathbf{r}''))\rho(\mathbf{r}',\mathbf{r}'',t). \tag{10}$$

The Hamiltonian (9) is deficient in one respect. It lacks an exchange potential.[6,11] The ultimate source of such a term lies in the antisymmetric character of many-electron wave functions. Guernsey,[25] von Roos,[6,27] von Roos and Zmuidzinas,[28] and Burt and Wahlquist[11] have discussed exchange effects for quantum plasmas and have argued or demonstrated the conditions under which they consititute small corrections. In particular, Burt and Wahlquist have shown (for the case of no external fields) that exchange results in only a minor correction to the dispersion relation in a quantum plasma.

In converting the density matrix equation into a kinetic equation we make one approximation, which is consistent with the neglect of exchange. The spin term involving the external magnetic field ($-\hat{\mu}\cdot\mathbf{B}$) does not survive (10) because $\mathbf{B}$ is spatially uniform. The self-consistent field term ($-\hat{\mu}\cdot\mathbf{B}_1$) is of course not spatially uniform and should be retained. It gives rise to a spin current and contributes to the conductivity and thus the dielectric tensor. Again we refer to the paper of Burt and Wahlquist who have shown that spin current corrections are small by comparison with exchange. Since the already minor effects of exchange are not considered in this report we also must drop the spin term. In passing we note that the disappearance of the external field spin term ($-\hat{\mu}\cdot\mathbf{B}$) from (10) *does not imply* that ρ is free of any spin dependence. It means only that the time variation of ρ is not subject to any such *direct* dependence. The spin dependence of ρ will enter via the equilibrium density matrix.

The definition of the Wigner function given by (3) is not sufficiently general to deal with a plasma in the presence of an electromagnetic field *if* $\mathbf{p}$ is identified as the *kinetic momentum*,

$$\mathbf{p}=M\mathbf{v}. \tag{11}$$

Instead, we introduce the wave vector $\mathbf{k}$, where $\hbar\mathbf{k}$ corresponds to the *canonical momentum*, viz.,

$$\hbar\mathbf{k}=\mathbf{p}+(e/c)\mathbf{A}(\mathbf{r},t), \tag{12}$$

with $\mathbf{p}$ given by (11). The appropriate Wigner function is defined by

$$f(\mathbf{r},\mathbf{p},t)=(\pi\hbar)^{-3}\int\exp(2i\mathbf{k}\cdot\boldsymbol{\lambda})\rho(\mathbf{r}-\boldsymbol{\lambda},\ \mathbf{r}+\boldsymbol{\lambda},\ t)d^3\lambda. \tag{13}$$

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Note that $\mathbf{k}$ is a function of $\mathbf{r}$ and t by virtue of its dependence on $\mathbf{A}(\mathbf{r},t)$,

$$2i\mathbf{k}\cdot\boldsymbol{\lambda}=\frac{2i}{\hbar}\mathbf{p}\cdot\boldsymbol{\lambda}+\frac{2ie}{\hbar c}\mathbf{A}(\mathbf{r},t)\cdot\boldsymbol{\lambda}. \tag{14}$$

The Wigner function satisfies a kinetic equation—the quantum version of the Boltzmann-Vlasov equation. To obtain the kinetic equation one takes

$$\mathbf{r}'=\mathbf{r}-\boldsymbol{\lambda},\quad \mathbf{r}''=\mathbf{r}+\boldsymbol{\lambda}$$

in (10), multiplies both sides by $(\pi\hbar)^{-3}\exp(2i\mathbf{k}\cdot\boldsymbol{\lambda})$, and integrates over λ. There ensues a straightforward, if wearisome, series of manipulations to obtain the kinetic equation for $f(\mathbf{r},\mathbf{p},t)$. At an intermediate stage one obtains[29]

$$\begin{aligned}
&\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla f+\frac{e}{c}(\mathbf{v}\times\mathbf{B})\cdot\nabla_p f+\frac{e}{Mc}\mathbf{A}_1\cdot\nabla f+\frac{e^2}{Mc^2}(\mathbf{A}_1\times\mathbf{B})\cdot\nabla_p f\\
&\quad+\frac{e}{c}\left\{\left(\mathbf{v}+\frac{e}{Mc}\mathbf{A}_1\right)\times\mathbf{B}_1\cdot\nabla_p f-\left(\mathbf{v}+\frac{e}{Mc}\mathbf{A}_1\right)\cdot(\nabla_p f\cdot\nabla)\mathbf{A}_1(\mathbf{r})\right\}-\frac{e}{c}\frac{\partial\mathbf{A}_1}{\partial t}\cdot\nabla_p f-\frac{e}{2Mc}\int\{\mathbf{A}_1(\mathbf{r}-\boldsymbol{\lambda})+\mathbf{A}_1(\mathbf{r}+\boldsymbol{\lambda})\}\\
&\quad\cdot(\nabla\rho)\exp(2i\mathbf{k}\cdot\boldsymbol{\lambda})d^3\lambda-\frac{e^2}{4Mc^2}\mathbf{B}\times\left(\nabla_p\int\rho\exp(2i\mathbf{k}\cdot\boldsymbol{\lambda})\right)\cdot\{\mathbf{A}_1(\mathbf{r}-\boldsymbol{\lambda})+\mathbf{A}_1(\mathbf{r}+\boldsymbol{\lambda})\}d^3\lambda\\
&=-\frac{ie}{\hbar}\int\{\varphi(\mathbf{r}-\boldsymbol{\lambda})-\varphi(\mathbf{r}+\boldsymbol{\lambda})\}\rho\exp(2i\mathbf{k}\cdot\boldsymbol{\lambda})d^3\lambda\\
&\qquad+\frac{ie}{\hbar c}\left(\mathbf{v}+\frac{e}{Mc}\mathbf{A}_1(\mathbf{r})\right)\cdot\int\{\mathbf{A}_1(\mathbf{r}-\boldsymbol{\lambda})-\mathbf{A}_1(\mathbf{r}+\boldsymbol{\lambda})\}\rho\exp(2i\mathbf{k}\cdot\boldsymbol{\lambda})d^3\lambda.
\end{aligned}\tag{15}$$

This equation is linearized by writing

$$f(\mathbf{r},\mathbf{p},t)=f_0(\mathbf{p})+f_1(\mathbf{r},\mathbf{p},t) \tag{16}$$

and treating $f_1(\mathbf{r},\mathbf{p},t)$, $\mathbf{A}_1(\mathbf{r},t)$, and $\varphi(\mathbf{r},t)$ as small quantities, rejecting all terms which are of second order. The only term of zero order is

$$(e/c)(\mathbf{v}\times\mathbf{B})\cdot\nabla_p f_0.$$

Equating this with zero imposes the condition (see Fig. 1)

$$\partial f_0/\partial\phi=0,$$

which is satisfied if

$$f_0(\mathbf{p})=f_0(p_\perp,p_\parallel). \tag{17}$$

Thus the equilibrium Wigner function must be cylindrically symmetric in momentum space, the axis of symmetry being defined by the external magnetic field. This same requirement is encountered in the classical theory.[30] The nonequilibrium portion of the Wigner function satisfies

$$\begin{aligned}
&\frac{\partial f_1}{\partial t}+\mathbf{v}\cdot\nabla f_1+\frac{e}{c}(\mathbf{v}\times\mathbf{B})\cdot\nabla_p f_1-\frac{e}{c}\frac{\partial\mathbf{A}_1}{\partial t}\cdot\nabla_p f_0\\
&\quad+\frac{e^2}{Mc^2}(\mathbf{A}_1\times\mathbf{B})\cdot\nabla_p f_0-\frac{e}{2Mc}\int\{\mathbf{A}_1(\mathbf{r}-\boldsymbol{\lambda})+\mathbf{A}_1(\mathbf{r}+\boldsymbol{\lambda})\}\cdot(\nabla\rho)\exp(2i\mathbf{k}\cdot\boldsymbol{\lambda})d^3\lambda\\
&\quad-\frac{e^2}{4Mc^2}\mathbf{B}\times\left(\nabla_p\times\int\rho\exp(2i\mathbf{k}\cdot\boldsymbol{\lambda})\right)\cdot\{\mathbf{A}_1(\mathbf{r}-\boldsymbol{\lambda})+\mathbf{A}_1(\mathbf{r}+\boldsymbol{\lambda})\}d^3\lambda+\frac{e}{c}\{(\mathbf{v}\times\mathbf{B}_1)\cdot\nabla_p f_0-\mathbf{v}\cdot(\nabla_p f_0\cdot\nabla)\mathbf{A}_1(\mathbf{r})\}\\
&=-\frac{ie}{\hbar}\int\left\{\varphi(\mathbf{r}-\boldsymbol{\lambda})-\varphi(\mathbf{r}+\boldsymbol{\lambda})-\frac{\mathbf{v}}{c}\cdot[\mathbf{A}_1(\mathbf{r}-\boldsymbol{\lambda})-\mathbf{A}_1(\mathbf{r}+\boldsymbol{\lambda})]\right\}\rho\exp(2i\mathbf{k}\cdot\boldsymbol{\lambda})d^3\lambda.
\end{aligned}\tag{18}$$

Expanding $\mathbf{A}_1(\mathbf{r}\pm\boldsymbol{\lambda})$ and $\varphi(\mathbf{r}\pm\boldsymbol{\lambda})$ in Taylor series and retaining only the first nonvanishing terms we get

$$\begin{aligned}(\partial f_1/\partial t)+\mathbf{v}\cdot\nabla f_1+(e/c)(\mathbf{v}\times\mathbf{B})\cdot\nabla_p f_1\\
=-e[\mathbf{E}_1+(1/c)(\mathbf{v}\times\mathbf{B}_1)]\cdot\nabla_p f_0,\end{aligned}\tag{19}$$

which is just the linearized Boltzmann-Vlasov equation. It should *not* be concluded that (19) is the "classical

[29] The factor $(\pi\hbar)^{-3}$ appearing in the definition of $f(\mathbf{r},\mathbf{p},t)$ is suppressed throughout this section.

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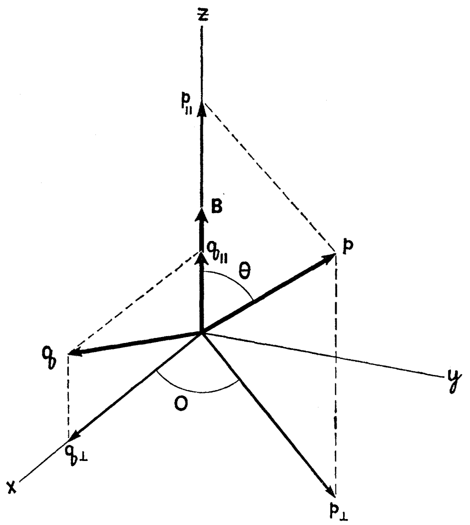

FIG. 1. Geometry for wave vector and momentum.

limit" ($\hbar=0$) of (18) since there is a considerable amount of quantum mechanics hidden in $f_0(p_\perp,p_{\shortparallel})$. The terms omitted by the abbreviated power-series expansion contribute terms to the dielectric tensor[31] which are of the same order as those lost by neglecting the spin current term $(-\hat{\mu}\cdot\mathbf{B}_1)$.

III. DIELECTRIC TENSOR

In order to determine the conductivity or dielectric tensors we require only the Fourier transform of $f_1(\mathbf{r},\mathbf{p},t)$. This is most readily obtained by working directly with the Fourier transformed equation. We write

$$\begin{Bmatrix} f_1(\mathbf{r},\mathbf{p},t) \\ \mathbf{E}_1(\mathbf{r},t) \\ \mathbf{B}_1(\mathbf{r},t)\end{Bmatrix} = \int \begin{Bmatrix} G(\mathbf{q},\mathbf{p},\omega) \\ \mathfrak{E}(\mathbf{q},\omega) \\ \mathfrak{B}(\mathbf{q},\omega)\end{Bmatrix} e^{i(\mathbf{q}\cdot\mathbf{r}+\omega t)}d^3q d\omega. \quad (20)$$

The transform $G(\mathbf{q},\mathbf{p},\omega)$ satisfies

$$(\omega+\mathbf{q}\cdot\mathbf{v})G+i\omega_c\frac{\partial G}{\partial\phi} = \frac{ie}{\omega}[(\omega+\mathbf{q}\cdot\mathbf{v})\mathfrak{E}-(\mathbf{v}\cdot\mathfrak{E})\mathbf{q}]\cdot\nabla_p f_0. \quad (21)$$

The cyclotron frequency is denoted as $\omega_c=eB/Mc$. One can solve (21) by the standard method of introducing an integrating factor. It is much more expedient to use the integral transformation introduced by Oberman and Ron.[7] We write

$$G(\mathbf{q},\mathbf{p},\omega)=e^{ib\sin\phi}\sum_{m=-\infty}^{+\infty} e^{-im\phi}J_m(b)G^{(m)}(p_\perp,p_{\shortparallel},\mathbf{q},\omega). \quad (22)$$

[31] In general, the same remark applies for all transport coefficients. More detailed comment concerning the neglected terms will be found in Sec. V.

$J_m(b)$ is a Bessel function of order m and

$$b=q_\perp p_\perp/M\omega_c. \quad (23)$$

$q_\perp$ and $q_{\shortparallel}$ denote the components of the wave vector which are perpendicular and parallel to the external magnetic field. Figure 1 illustrates the geometry. The wave vector may be taken to lie in the $x-z$ plane,

$$\mathbf{q}=(q_\perp,0,q_{\shortparallel}), \quad (24)$$

and cylindrical coordinates are appropriate for the momentum variable,

$$\mathbf{p}=(p_\perp\cos\phi,p_\perp\sin\phi,p_{\shortparallel}). \quad (25)$$

The transformation, (22), splits off the ϕ dependence of $G(\mathbf{p},\mathbf{q},\omega)$ and converts (21) into a set of algebraic equations for $J_m(b)G^{(m)}(p_\perp,p_{\shortparallel},\mathbf{q},\omega)$. Using (22) in (21) results in

$$e^{ib\sin\phi}\sum_{m=-\infty}^{+\infty}(\omega+q_{\shortparallel}v_{\shortparallel}+m\omega_c)e^{-im\phi}J_m(b)G^{(m)} = \frac{ie}{\omega}[(\omega+\mathbf{q}\cdot\mathbf{v})\mathfrak{E}-(\mathbf{v}\cdot\mathfrak{E})\mathbf{q}]\cdot\nabla_p f_0.$$

Multiplying through by $(2\pi)^{-1}e^{-i(b\sin\phi-n\phi)}$ and integrating gives

$$(\omega+q_{\shortparallel}v_{\shortparallel}+n\omega_c)J_n(b)G^{(n)} = \frac{ie}{\omega}\frac{1}{2\pi}\int_0^{2\pi} e^{-i(b\sin\phi-n\phi)} \times d\phi[(\omega+\mathbf{q}\cdot\mathbf{v})\mathfrak{E}-(\mathbf{v}\cdot\mathfrak{E})\mathbf{q}]\cdot\nabla_p f_0. \quad (26)$$

The integration on the right side results in a bevy of Bessel functions. Having obtained $J_n(b)G^{(n)}$ in this fashion one can reconstruct $G(\mathbf{p},\mathbf{q},\omega)$ via (20).

The current density induced by the self-consistent fields is given by

$$\mathbf{J}(\mathbf{r},t)=\frac{e}{M}\int \mathbf{p}f_1(\mathbf{r},\mathbf{p},t)d^3p. \quad (2)$$

The fourier transform of $\mathbf{J}(\mathbf{r},t)$, denoted as $\mathfrak{J}(\mathbf{q},\omega)$, is given by

$$\mathfrak{J}(\mathbf{q},\omega)=\frac{e}{M}\int \mathbf{p}G(\mathbf{p},\mathbf{q},\omega)d^3p. \quad (27)$$

For the infinite medium considered here, the relation between $\mathfrak{J}(\mathbf{q},\omega)$ and $\mathfrak{E}(\mathbf{q},\omega)$ defines a conductivity tensor $\boldsymbol{\sigma}(\mathbf{q},\omega)$,

$$\mathfrak{J}(\mathbf{q},\omega)=\boldsymbol{\sigma}\cdot\mathfrak{E}(\mathbf{q},\omega). \quad (28)$$

Inserting (22) in (27) allows one to perform the ϕ integration in (27) and results in more of the ever-ready Bessel functions. The dielectric tensor $\boldsymbol{\epsilon}(\mathbf{q},\omega)$ links the Fourier transform of the electric induction $\mathfrak{D}(\mathbf{q},\omega)$ to $\mathfrak{E}(\mathbf{q},\omega)$.

$$\mathfrak{D}(\mathbf{q},\omega)\equiv\boldsymbol{\epsilon}\cdot\mathfrak{E}(\mathbf{q},\omega). \quad (29)$$

In terms of $\boldsymbol{\sigma}$,

$$\epsilon(\mathbf{q},\omega)=\mathbf{I}-\frac{4\pi i}{\omega}\boldsymbol{\sigma}(\mathbf{q},\omega). \tag{30}$$

Routine calculation leads to the result (ω_p=plasma frequency)

$$\epsilon_{\alpha\beta}=\left(1-\frac{\omega_p^2}{\omega^2}\right)\delta_{\alpha\beta}-\frac{\omega_p^2}{\omega^2}n_{\alpha\beta}. \tag{31}$$

If we introduce the notation

$$f_{\parallel}\equiv\frac{\partial f_0}{\partial p_{\parallel}},\quad f_{\perp}\equiv\frac{\partial f_0}{\partial p_{\perp}},$$
$$S[x]\equiv\sum_{n=-\infty}^{+\infty}\int\frac{d^3p[x]}{\omega+q_{\parallel}v_{\parallel}+n\omega_c}, \tag{32}$$

the elements of $\mathfrak{n}(\mathbf{q},\omega)$ may be written down. (The Wigner function is now normalized to unity.)

$$n_{11}=S\left[\left(\frac{nJ_n}{b}\right)^2 p_{\perp}\{v_{\perp}q_{\parallel}f_{\parallel}+n\omega_c f_{\perp}\}\right], \tag{33}$$

$$n_{12}=-n_{21}=S\left[i\left(\frac{nJ_nJ_n'}{b}\right)p_{\perp}\{v_{\perp}q_{\parallel}f_{\parallel}+n\omega_c f_{\perp}\}\right], \tag{34}$$

$$n_{13}=S\left[\frac{nJ_n^2}{b}p_{\perp}\left\{v_{\parallel}q_{\parallel}f_{\parallel}+\frac{v_{\parallel}n\omega_c}{v_{\perp}}f_{\perp}\right\}\right], \tag{35}$$

$$n_{22}=S[(J_n')^2p_{\perp}\{v_{\perp}q_{\parallel}f_{\parallel}+n\omega_c f_{\perp}\}], \tag{36}$$

$$n_{23}=S\left[-iJ_nJ_n'p_{\perp}\left\{v_{\parallel}q_{\parallel}f_{\parallel}+\frac{v_{\parallel}n\omega_c}{v_{\perp}}f_{\perp}\right\}\right], \tag{37}$$

$$n_{31}=S\left[\frac{nJ_n^2}{b}p_{\parallel}\{v_{\perp}q_{\parallel}f_{\parallel}+n\omega_c f_{\perp}\}\right], \tag{38}$$

$$n_{32}=S[iJ_nJ_n'p_{\parallel}\{v_{\perp}q_{\parallel}f_{\parallel}+n\omega_c f_{\perp}\}], \tag{39}$$

$$n_{33}=S\left[J_n^2p_{\parallel}\left\{v_{\parallel}q_{\parallel}f_{\parallel}+\frac{v_{\parallel}n\omega_c}{v_{\perp}}f_{\perp}\right\}\right]. \tag{40}$$

The dielectric tensor will be evaluated for several cases of interest in Sec. V. In passing we note that the dispersion relation may be expressed in terms of $n_{\alpha\beta}$ as

$$\mathrm{Det}\left[\left(q^2-\frac{\omega^2-\omega_p^2}{c^2}\right)\delta_{\alpha\beta}-q_\alpha q_\beta+\frac{\omega_p^2}{c^2}n_{\alpha\beta}\right]=0. \tag{41}$$

Alternately we may write

$$\mathrm{Det}[c^2(q^2\delta_{\alpha\beta}-q_\alpha q_\beta)-\omega^2\epsilon_{\alpha\beta}]=0. \tag{41a}$$

IV. EQUILIBRIUM WIGNER FUNCTION

In order to evaluate $\mathfrak{n}(\mathbf{q},\omega)$, one requires the equilibrium Wigner function $f_0(\mathbf{p})$. The equilibrium density matrix is

$$\rho(\mathbf{r}',\mathbf{r}'')=\textstyle\sum_j W_ju_j^*(\mathbf{r}'')u_j(\mathbf{r}'), \tag{42}$$

where W_j is the occupation probability for a state of energy E_j. The u_j are energy eigenfunctions of the time-independent Schrödinger equation. We consider two cases:

$$W_j=e^{-\beta(E_j-\mu)},\quad\text{(Maxwell-Boltzmann)} \tag{43}$$

$$W_j=\{1+e^{\beta(E_j-\mu)}\}^{-1},\quad\text{(Fermi-Dirac)} \tag{44}$$

where

$$\beta^{-1}=k_BT.$$

In (43) and (44), $\mu=\mu(\beta)$ is the chemical potential and is determined by the normalization restriction

$$\int f_0(\mathbf{p})d^3p=1. \tag{45}$$

The equilibrium Wigner function is given by

$$f_0(\mathbf{p})=(\pi\hbar)^{-3}\int\rho(\mathbf{r}-\boldsymbol{\lambda},\,\mathbf{r}+\boldsymbol{\lambda})\times\exp[(2i/\hbar)(\mathbf{p}+(e/c)\mathbf{A}_0(\mathbf{r}))\cdot\boldsymbol{\lambda}]d^3\lambda. \tag{46}$$

The details of the calculation are given in the Appendix. For Maxwell-Boltzmann statistics one has

$$f_0(p_{\perp},p_{\parallel})=\frac{\tanh\Theta}{M\hbar\omega_c}\left(\frac{\beta}{2M\pi^3}\right)^{1/2}\times\exp\left(-\frac{\beta p_{\parallel}^2}{2M}-\tanh\Theta\frac{p_{\perp}^2}{M\hbar\omega_c}\right), \tag{47}$$

with

$$\Theta=\tfrac{1}{2}\beta\hbar\omega_c.$$

When $\Theta\ll1$ (high temperatures and/or weak field), $\tanh\Theta/M\hbar\omega_c\approx\beta/2M$ and (47) reduces to the familiar Maxwellian form. For Fermi-Dirac statistics one finds

$$f_0(p_{\perp},p_{\parallel})=\frac{2e^{-w^2}}{N_e(2\pi\hbar)^3}\sum_{n=0,\sigma=\pm\frac{1}{2}}^{\infty}\frac{(-1)^nL_n(2w^2)}{[1+\exp\beta\{(p_{\parallel}^2/2M)+\hbar\omega_c(n+\sigma+\frac{1}{2})-\mu\}]}, \tag{48}$$

with N_e denoting the electron concentration and

$$w^2=p_{\perp}^2/M\hbar\omega_c. \tag{49}$$

$L_n(2w^2)$ is the Laguerre polynomial[32] of order n. The chemical potential is determined by the normalization condition. Using the result

$$2(-1)^n\int_0^\infty e^{-w^2}L_n(2w^2)wdw=1\,,$$

gives

$$N_e=\frac{M\omega_c}{(2\pi\hbar)^2}\sum_{n=0,\sigma=\pm\frac{1}{2}}^{\infty}\int_{-\infty}^{+\infty}\frac{dp_{\parallel}}{[1+\exp\beta(\{p_{\parallel}^2/2M)+\hbar\omega_c(n+\sigma+\frac{1}{2})-\mu\}]}\,. \tag{50}$$

At $T=0°$K ($\beta=\infty$), the integration may be performed since the distribution over $p_{\parallel}$ constitutes a "Landau ladder." The result of the integration is

$$N_e=\frac{2^{3/2}M\omega_c}{(2\pi\hbar)^2}\sum_{n=0}^{n}\alpha_n(\mu-n\hbar\omega_c)^{1/2}\,, \tag{51}$$

with

$$\alpha_0=1\,,\quad \alpha_{n>0}=2\,. \tag{52}$$

The summation of (51) extends over all n for which $\mu-n\hbar\omega_c$ is positive.[33] In the strong field limit

$$\hbar\omega_c>\mu\,,$$

only the $n=0$ term emerges and (51) yields the Fermi energy

$$\mu=\frac{2\pi^4N_e^2}{(\hbar\omega_c)^2}\left(\frac{\hbar^2}{M}\right)^3;\quad \hbar\omega_c>\mu\,. \tag{53}$$

In the opposite extreme of zero field, the summation is converted to an integral which gives the familiar result

$$\mu=\frac{2\pi\hbar^2}{M}\left(\frac{3N_e}{8\pi}\right)^{2/3};\quad \hbar\omega_c\approx 0\,. \tag{54}$$

Inspection of (48) suggests that the equilibrium Wigner function is capable of describing effects associated with the periodicity of the density of states.[34] Further comment on this point is postponed. In the next section we evaluate the dielectric tensor for several cases of interest.

V. EVALUATION OF THE DIELECTRIC TENSOR

In this section we present results for the dielectric tensor for waves propagating along and across the magnetic field.

A. Longitudinal Propagation: $q_\perp=0$, $q_{\parallel}=q$

In this case one finds

$$n_{11}=n_{22}=n_++n_-\,,$$
$$n_{12}=-n_{12}=i(n_+-n_-)\,,$$

where

$$n_\pm=\frac{1}{4}\int p_\perp\left(\frac{qv_\perp f_{\parallel}\pm\omega_c f_\perp}{\omega\pm\omega_c+qv_{\parallel}}\right)d^3p\,. \tag{55}$$

The only other nonzero element of $\mathfrak{n}(\mathbf{q},\omega)$ is

$$n_{33}=\frac{q}{M}\int\frac{p_{\parallel}^2f_{\parallel}d^3p}{\omega+qv_{\parallel}}\,. \tag{56}$$

The dispersion relation, (41a), becomes

$$\omega^2=\omega_p^2(1+n_{33})\,, \tag{57}$$
$$\omega^2=q^2c^2+\omega_p^2(1+2n_\pm)\,. \tag{58}$$

Equation (57) describes the longitudinal waves while (58) governs the circularly polarized transverse waves. For Boltzmann statistics, the long-wavelength approximations for n_{33} and $2n_\pm$ are

$$2n_\pm=\frac{\mp\omega_c}{\omega\pm\omega_c}+\frac{q^2}{M\beta(\omega\pm\omega_c)^2}\left\{\Theta\coth\Theta\mp\frac{\omega_c}{\omega\pm\omega_c}\right\}\,, \tag{55MB}$$
$$n_{33}=3q^2/M\beta\omega^2\,. \tag{56MB}$$

When $\Theta\ll 1$, $\Theta\coth\Theta\approx 1$ and $2n_\pm$ reduces to the usual classical result. The corresponding dispersion relations read

$$\omega^2=\omega_p^2+3q^2/M\beta\,, \tag{57MB}$$

$$\omega^2=q^2c^2+\frac{\omega_p^2\omega}{\omega\pm\omega_c}+\frac{q^2\omega_p^2}{M\beta(\omega\pm\omega_c)^2}\times\left\{\Theta\coth\Theta\mp\frac{\omega_c}{\omega\pm\omega_c}\right\}\,. \tag{58MB}$$

For Fermi-Dirac statistics, the integrations can be performed at $T=0°$K ($\beta=\infty$) and give

$$2n_\pm=\left(\frac{q^2\hbar\omega_c}{2M}\sum_{n=0}^{n_F}\frac{\alpha_nV_n}{(\omega\pm\omega_c)^2-q^2V_n^2}\mp\frac{\omega_c}{q}\sum_{n=0}^{n_F}\alpha_n\ln\left|\frac{\omega\pm\omega_c+qV_n}{\omega\pm\omega_c-qV_n}\right|\right)\Bigg/\sum_{n=0}^{n_F}\alpha_nV_n\,, \tag{55FD}$$

$$n_{33}=\frac{q^2\mu}{M\omega^2}\chi\left(\frac{\hbar\omega_c}{\mu}\right)\,. \tag{56FD}$$

[32] G. Sansone, *Orthogonal Functions* (Interscience Publishers, Inc., New York, 1959).
[33] The normalization condition, (51), is readily identified as the usual quantum "sum over states."
[34] M. Dresden, Rev. Mod. Phys. **33**, 265 (1961).

The function $\chi(\hbar\omega_c/\mu)$ is defined as

$$\chi\left(\frac{\hbar\omega_c}{\mu}\right)=\sum_{n=0}^{n_F}\alpha_n\left(1-\frac{\hbar\omega_c}{\mu}n\right)^{3/2}\Bigg/\sum_{n=0}^{n_F}\alpha_n\left(1-\frac{\hbar\omega_c}{\mu}n\right)^{1/2}. \quad (59)$$

V_n is the maximum speed along the direction of the field for electrons on the nth "rung" of the Landau ladder,

$$\tfrac{1}{2}MV_n^2=\mu-n\hbar\omega_c. \quad (60)$$

In arriving at these results we have taken Cauchy principle values and ignored the imaginary contributions. In the strong field realm ($\hbar\omega_c>\mu$, only $n=0$, $\sigma=-\frac{1}{2}$ level populated) these become

$$n_{33}=2q^2\mu/M\omega^2,$$

$$2n_\pm=\frac{q^2\hbar\omega_c}{2M}\frac{1}{(\omega\pm\omega_c)^2-q^2V_0^2}\mp\frac{\omega_c}{2qV_0}\ln\left|\frac{\omega\pm\omega_c+qV_0}{\omega\pm\omega_c-qV_0}\right|.$$

B. Transverse Propagation: $q_\perp=q$, $q_\parallel=0$

For waves propagating across the magnetic field the nonzero elements of $\mathfrak{n}(\mathbf{q},\omega)$ are n_{11}, n_{12}, n_{21}, n_{22}, and n_{33}. The long-wavelength results for the dielectric tensor in the case of Maxwell-Boltzmann statistics are

$$\epsilon_{11}=1-\frac{\omega_p^2}{\omega^2-\omega_c^2}+\frac{q^2\hbar\omega_c}{2M}\frac{\omega_p^2}{\omega^2}(\Delta_1-4\Delta_2)\coth\Theta, \quad (61\text{MB})$$

$$\epsilon_{12}=-\epsilon_{21}=\frac{i\omega\omega_c}{\omega^2-\omega_c^2}\frac{\omega_p^2}{\omega^2}-\frac{iq^2\hbar\omega_c}{M}\frac{\omega_p^2}{\omega^2}\times(\Delta_1-\Delta_2)\coth\Theta, \quad (62\text{MB})$$

$$\epsilon_{22}=1-\frac{\omega_p^2}{\omega^2-\omega_c^2}+\frac{q^2\hbar\omega_c}{2M}\frac{\omega_p^2}{\omega^2}(3\Delta_1-4\Delta_2)\coth\Theta, \quad (63\text{MB})$$

$$\epsilon_{33}=1-\frac{\omega_p^2}{\omega^2}\left(1+\frac{q^2}{M\beta(\omega^2-\omega_c^2)}\right), \quad (64\text{MB})$$

where

$$\Delta_n=1/(\omega^2-n^2\omega_c^2). \quad (65)$$

In the case of Fermi-Dirac statistics at $T=0$°K, the form of the dielectric tensor appears quite similar to the Maxwell-Boltzmann version. One finds

$$\epsilon_{11}=1-\frac{\omega_p^2}{\omega^2-\omega_c^2}+\frac{q^2\hbar\omega_c}{2M}\frac{\omega_p^2}{\omega^2}(\Delta_1-4\Delta_2)\Psi\left(\frac{\hbar\omega_c}{\mu}\right), \quad (61\text{FD})$$

$$\epsilon_{12}=-\epsilon_{21}=\frac{i\omega\omega_c}{\omega^2-\omega_c^2}\frac{\omega_p^2}{\omega^2}-\frac{iq^2\hbar\omega_c}{M}\frac{\omega_p^2}{\omega^2}(\Delta_1-\Delta_2)\Psi\left(\frac{\hbar\omega_c}{\mu}\right), \quad (62\text{FD})$$

$$\epsilon_{22}=1-\frac{\omega_p^2}{\omega^2-\omega_c^2}+\frac{q^2\hbar\omega_c}{2M}\frac{\omega_p^2}{\omega^2}(3\Delta_1-4\Delta_2)\Psi\left(\frac{\hbar\omega_c}{\mu}\right), \quad (63\text{FD})$$

$$\epsilon_{33}=1-\frac{\omega_p^2}{\omega^2}\left(1+\frac{2q^2\mu}{3M(\omega^2-\omega_c^2)}\chi\left(\frac{\hbar\omega_c}{\mu}\right)\right), \quad (64\text{FD})$$

where

$$\Psi\left(\frac{\hbar\omega_c}{\mu}\right)=2\sum_{n=0}^{n_F}\alpha_n n\left(1-\frac{\hbar\omega_c}{\mu}n\right)^{1/2}\Bigg/\sum_{n=0}^{n_F}\alpha_n\left(1-\frac{\hbar\omega_c}{\mu}n\right)^{1/2}. \quad (66)$$

The long-wavelength results for the dielectric tensor at $T=0$°K are in complete agreement with the recent unpublished results of Stephen, who employed the field theoretic techniques of Ref. 13. In the limit of zero field, the sums in Ψ and χ may be replaced by integrals. One finds

$$\hbar\omega_c\Psi\left(\frac{\hbar\omega_c}{\mu}\right)\Bigg|_{\hbar\omega_c=0}=\tfrac{4}{5}\mu, \quad (67)$$

$$\chi(0)=\tfrac{3}{5}. \quad (68)$$

In this limit the dielectric tensor is diagonal,

$$\epsilon_{11}=1-\frac{\omega_p^2}{\omega^2}\left(1+\frac{6q^2\mu}{5M\omega^2}\right), \quad (69)$$

$$\epsilon_{22}=\epsilon_{33}=1-\frac{\omega_p^2}{\omega^2}\left(1+\frac{2q^2\mu}{5M\omega^2}\right). \quad (70)$$

These results agree with those of Lindhard.[4]

We have remained oblivious of the fact that Eq. (19), leading to the results of this section, is in fact a long-wavelength approximation of the true kinetic equation. It is not much more difficult to carry through an exact analysis. When this is done, one finds that the deviations from the results quoted here involve terms of order $\hbar^2q^2/M\mu$ times the "q^2 terms" in $\epsilon(\mathbf{q},\omega)$. To put it another way, if we let q_F denote the wave vector of an electron on the Fermi sphere, the neglected terms are of order $(q/q_F)^2$ times the smallest terms retained. The neglect of such terms is justified within the framework of the self-consistent field. The spirit of the self-consistent field is that the collective effect of many particles dominates any short-range correlations. The scale of the collective interaction is $1/q$; that of the correlations, $1/q_F$. Thus, the self-consistent field becomes an inappropriate vehicle for describing a quantum plasma when $(q/q_F)^2$ becomes comparable with unity.

VI. CONCLUSION

The results obtained for the dielectric tensor indicate that the Wigner function is an appropriate quantum distribution function, even though it lacks the right to be termed a probability density in phase space. Although we have treated the fields $\mathbf{E}_1$ and $\mathbf{B}_1$ (or their transforms $\mathfrak{E}$ and $\mathfrak{B}$) as agents describing the mutual interaction of the plasma constituents the same *linearized* Boltzmann-Vlasov equation, (19), would apply if $\mathbf{E}_1$ and $\mathbf{B}_1$ described an *external* electromagnetic field. The natural extension is to introduce the effect of binary collisions through a relaxation term similar to that used by Karplus and Schwinger[35]

$$(\partial f_1/\partial t)_{\text{collision}} = -\nu f_1(\mathbf{r},\mathbf{p},t)$$

and write the quantum kinetic equation for the perturbed segment of the Wigner function as

$$\frac{\partial f_1}{\partial t}+\mathbf{v}\cdot\nabla f_1+\frac{e}{c}(\mathbf{v}\times\mathbf{B})\cdot\nabla_p f_1+\nu f_1 = -e\left(\mathbf{E}_1+\frac{1}{c}\mathbf{v}\times\mathbf{B}_1\right)\cdot\nabla_p f_0, \quad (19a)$$

with $f_0(\mathbf{p})$ denoting the equilibrium Wigner function. It is suggested that such an equation constitutes the basis for a relatively simple "quantized kinetic theory" capable of handling problems heretofore explored only by more sophisticated means. Thus, for example, it seems likely that many phenomena which depend on the periodicity of the density of states (e.g., the de Haas-van Alphen effect, the Shubnikov-de Haas effect, and magnetothermal oscillations[36]) may be studied quantitatively within the "pedestrian" framework of kinetic theory.

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APPENDIX

The energy eigenfunctions for a charged particle in a uniform magnetic field are

$$u_{n,k_1,k_3}(\mathbf{r}) = \left(\frac{M\omega_c}{\pi\hbar}\right)^{1/4}(2^n n! L_x L_z)^{-1/2} \times e^{i(k_1x+k_3z)}e^{-\frac{1}{2}u^2}H_n(u), \quad (A1)$$

with

$$u^2 = (M\omega_c/\hbar)(y+(\hbar k_1/M\omega_c))^2. \quad (A2)$$

$H_n(u)$ is the Hermite polynomial[32] of order n. The energy spectrum is

$$E_{n,k_3,\sigma} = (\hbar^2k_3^2/2M)+\hbar\omega_c(n+\sigma+\tfrac{1}{2}), \quad (A3)$$

where the spin index σ takes on the values $\pm\frac{1}{2}$. The vector potential corresponding to the eigenfunctions is[37]

$$\mathbf{A}_0(\mathbf{r}) = (-By, 0, 0),$$

so that

$$\frac{2i}{\hbar}\left(\mathbf{p}+\frac{e}{c}\mathbf{A}_0\right)\cdot\boldsymbol{\lambda} = \frac{2i}{\hbar}[(p_x-M\omega_c y)\lambda_x+p_y\lambda_y+p_z\lambda_z]. \quad (A4)$$

For Boltzmann statistics the Wigner function is

$$f_0(\mathbf{p}) = \frac{(\pi\hbar)^{-3}}{L_xL_z}\left(\frac{M\omega_c}{\pi\hbar}\right)^{1/2}\sum_{n,k_1,k_3,\sigma}\frac{1}{2^n n!}\iiint_{-\infty}^{+\infty}d\lambda_x d\lambda_y d\lambda_z \exp\left[-2i\lambda_x\left(k_1+\frac{M\omega_c}{\hbar}y-\frac{p_x}{\hbar}\right)+\frac{2i\lambda_y}{\hbar}p_y-2i\lambda_z\left(k_3-\frac{p_{\parallel}}{\hbar}\right)\right]$$
$$\times\exp\beta\left[\mu-\frac{\hbar^2k_3^2}{2M}-\hbar\omega_c(n+\sigma+\tfrac{1}{2})\right]e^{-\frac{1}{2}[(u+v)^2+(u-v)^2]}H_n(u+v)H_n(u-v), \quad (A5)$$

where

$$v^2 \equiv (M\omega_c/\hbar)\lambda_y^2. \quad (A6)$$

The integrations over λ_x and λ_y introduce delta functions. Thus, for example,

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}e^{-2i\lambda_z(k_3-p_{\parallel}/\hbar)}d\lambda_z = \delta(k_3-p_{\parallel}/\hbar).$$

The sum over spins $(\sigma=\pm\frac{1}{2})$ gives a factor

$$e^{\Theta}+e^{-\Theta},$$

with

$$\Theta = \tfrac{1}{2}\beta\hbar\omega_c. \quad (A7)$$

Thus the Wigner function becomes

$$f_0(\mathbf{p}) = \frac{e^{\beta\mu}(1+e^{-2\Theta})}{L_xL_z(\pi\hbar^2)^{3/2}}\sum_{n,k_1,k_3}\frac{e^{-2n\Theta}}{2^n n!}\delta(k_3-p_{\parallel}/\hbar)\,e^{-\beta(\hbar^2k_3^2/2M)}\int_{-\infty}^{+\infty}\delta\left(k_1+\frac{M\omega_c}{\hbar}y-\frac{p_x}{\hbar}\right)$$
$$\times\exp\left\{\frac{2ip_yv}{(M\hbar\omega_c)^{1/2}}\right\}e^{-\frac{1}{2}[(u+v)^2+(u-v)^2]}H_n(u+v)H_n(u-v)dv.$$

[35] R. Karplus and J. Schwinger, Phys. Rev. **73**, 1020 (1948).
[36] J. E. Kunzler, F. S. L. Hsu, and W. S. Boyle, Phys. Rev. **128**, 1084 (1963).
[37] This $\mathbf{A}_0$ is not equal to $\frac{1}{2}\mathbf{B}\times\mathbf{r}$. This does not matter since $f_0(\mathbf{p})$ is a gauge-invariant quantity.

The sums over k_1 and k_3 are converted to integrals using

$$\sum_{k_1,k_3} = \frac{L_x L_z}{(2\pi)^2} \int\int_{-\infty}^{+\infty} dk_1 dk_3 .$$

This gives

$$f_0(\mathbf{p}) = \frac{2(1+e^{-2\Theta})}{(\pi)^{1/2}(2\pi\hbar)^3} e^{\beta[\mu-(p_{\parallel}^2/2M)]} \sum_{n=0}^{\infty} \frac{e^{-2n\Theta}}{2^n n!} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}[(p_1+v)^2+(p_1-v)^2]} e^{2ip_2 v} H_n(p_1-v) H_n(p_1+v) dv , \tag{A8}$$

where

$$p_1 = p_x/(M\hbar\omega_c)^{1/2}$$
$$p_2 = p_y/(M\hbar\omega_c)^{1/2} .$$

One can perform the summation using Mehler's formula, and then carry out the integration. However, this does not seem to be a useful technique when performing the corresponding calculations for Fermi-Dirac statistics. We therefore perform the integration, then the summation. The integral may be evaluated by using the generating function for the Hermite polynomials. The result is

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}[(p_1+v)^2+(p_1-v)^2]} e^{2ip_2 v} H_n(p_1-v) H_n(p_1+v) dv = (-1)^n 2^n n! (\pi)^{1/2} e^{-w^2} L_n(2w^2) , \tag{A9}$$

with

$$w^2 \equiv (p_x^2+p_y^2)/M\hbar\omega_c = p_\perp^2/M\hbar\omega_c . \tag{A10}$$

$L_n(2w^2)$ is a Laguerre polynomial of order n.[32] The summation over n may be performed using the generating function for Laguerre polynomials and yields

$$f_0(\mathbf{p}) = \frac{2e^{\beta\mu}}{(2\pi\hbar)^3} \exp\left(-\frac{\beta p_{\parallel}^2}{2M} - \tanh\Theta \frac{p_\perp^2}{M\hbar\omega_c} \right) .$$

The normalization condition requires

$$\int f_0 d^3p = 1 ,$$

and gives

$$f_0(p_\perp, p_{\parallel}) = \frac{\tanh\Theta}{M\hbar\omega_c} \left(\frac{\beta}{2M\pi^3} \right)^{1/2} \exp\left(-\frac{\beta p_{\parallel}^2}{2M} - \tanh\Theta \frac{p_\perp^2}{M\hbar\omega_c} \right) . \tag{A11}$$

The integrations for Fermi-Dirac statistics are the same but the spin sum and final sum over n are not performed. The normalized Wigner function for Fermi-Dirac statistics is

$$f_0(p_\perp, p_{\parallel}) = \frac{2e^{-w^2}}{N_e(2\pi\hbar)^3} \sum_{n=0,\sigma=\pm\frac{1}{2}}^{\infty} \frac{(-1)^n L_n(2w^2)}{[1+\exp\beta\{(p_{\parallel}^2/2M)+\hbar\omega_c(n+\sigma+\frac{1}{2})-\mu\}]} , \tag{A12}$$

where N_e is the electron concentration and w is defined by (A10).